# Nonlinear evolution of sand ripples in a viscous flow

A. Valance
*Institut de Physique de Rennes, UMR UR1-CNRS 6251, Université de Rennes 1, Campus Beaulieu, Bât 11A, 35042 Rennes Cedex, France*

ABSTRACT: We investigate theoretically the nonlinear evolution of sand ripples in a viscous fluid when a granular bed is submitted to a two-dimensional laminar shear flow. On the basis of the hydrodynamic equations and a semi-empirical sand transport law, we derive a non-local and nonlinear equation for the dynamics of the bed profile in the case of infinite flow depth. This equation reveals a coarsening process: the wavelength of the ripple pattern increases indefinitely with time and no final state is selected. The temporal evolution of the amplitude, wavelength and drift velocity of the ripple pattern is carefully analyzed: in particular, the drift speed is found inversely proportional to the ripple amplitude. Finally, we argue that coarsening may interrupt either when the flow is three-dimensional or when the flow depth is finite (i.e., comparable to the amplitude of the ripple pattern).

## 1 INTRODUCTION

The description and understanding of patterns in sand produced by the flow of air or water still challenges the community (Nishimori et al. 1993, Werner et al. 1993, Csahok et al. 2000, Betat et al. 1999, Stegner et al. 1999, Hansen et al. 2001, Andersen et al. 2002). Despite diverse efforts including phenomenological and stochastic models (Nishimori et al. 1993, Werner et al. 1993) for ripples in air, analogic nonlinear modeling for sand vortex ripples (Hansen et al. 2001, Andersen et al. 2002), or continuum description based on the symmetries of the problem (Csahok et al. 2000), theoretical understanding has remained sparse. Here, we address the issue of the wavelength selection of sand ripples in a steady and laminar two-dimensional liquid flow. A flat bed of sand subject to a steady fluid flow is generally unstable and gives rise to the so-called ripple patterns. In the first stages of the instability, the pattern presents a well-defined periodicity, but later on, the ripple wavelength tends to increase in course of time. In air, this coarsening process clearly stops probably due to the existence of a strong coupling between the sand transport and the shape of the sand bed. In water, this is still a matter of debate and there is no clear experimental evidence to provide for a definite answer (Baas 1994, Rehberg et al. 2002, Loiseleux et al. 2004). Some experiments (Baas 1994, Rehberg et al. 2002) show that the coarsening process stops while other ones (Loiseleux et al. 2002) suggest that the coarsening proceeds indefinitely. We show in this paper that in the case of two-dimensional liquid flow of infinite depth (i.e., much larger than the dimension of the ripple pattern) the ripple wavelength increases indefinitely in course of time and no final state is selected. We therefore argue that coarsening may be interrupted only when the flow is three-dimensional [as in (Baas 1994)] or when it is shallow [as in (Rehberg et al. 2002)].

On the basis of the hydrodynamic equations and a semi-empirical sand transport law, we are able to derive, in the case of a two-dimensional and steady liquid flow of infinite depth, a closed equation for the dynamics of the sand bed profile. This equation is non local and nonlinear and reveals the existence of a coarsening process that never stops. To our knowledge, this is the first time that such an equation is derived in the context of sand bedforms.

## 2 MODEL

We consider a Newtonian and viscous fluid flowing over a sand bed. We focus on a Couette flow configuration, i.e., the upper plate is pulled at an imposed and constant speed $U_0$ in the x-direction (see Fig. 1). The equations of motion for the fluid read:

$$\rho\,\partial_t u + \rho\,(u.\nabla u)\,u = -\nabla p + \eta\nabla^2 u \qquad (1)$$
$$\nabla.u = 0$$

**u**=(u,w) is the fluid velocity in the plane (O,x,z), p the pressure, ρ the volumetric mass of the fluid, and η its dynamic viscosity (ν=η/ρ being the kinematic viscosity). These equations should be supplemented with conditions at the boundaries. At the sand bed surface [i.e., z=h(x,t)], $- u\,\partial_x h + w = \partial_t h$ and $u+w\,\partial_x h = 0$ (the first condition expresses the continuity between the fluid velocity, perpendicular to the sand bed, and the normal displacement speed of the bed, and the second one is a consequence of the no-slip condition of the fluid at the sand bed surface). At the upper fluid surface (i.e., z=L), $u=U_0$ and w=0.

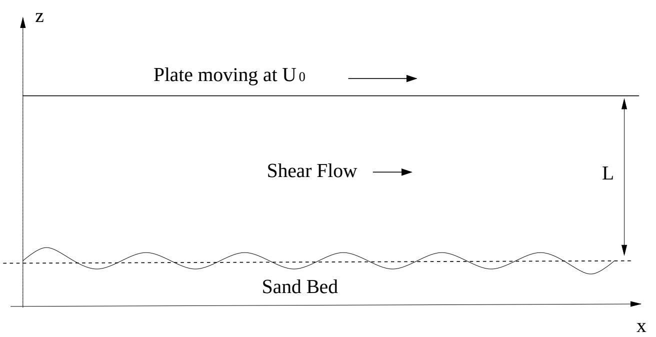


Fig.1: Two-dimensional laminar shear flow over a deformed sand bed.

The transport of sediment is induced by the bed shear stress (i.e., the flow shear stress calculated at the sand surface) but its precise evaluation is not a simple matter since it involves intricate and complex processes such as grain-grain and fluid-grain interactions. Up to now, there is no sound theoretical description for the transport of particles. Therefore, we will use the semi-empirical law established by Meyer-Peter and Müller (Fredsoe et al. 1992):

$$q = q_b\left[\Theta - \Theta_{c_0}\left(1-\frac{\tan\phi}{\tan\phi_s}\right)\right]^{3/2} \qquad (2)$$

Θ is the dimensionless bed shear stress, the so-called Shield parameter: $\Theta = \sigma/\rho g(s-1)d$, where σ is the bed shear stress, $s=\rho_g/\rho$ is the relative density of the sediment compared to that of the fluid, g the gravitational acceleration and d the diameter of the grains. ϕ is the angle of the local sand bed slope (i.e., $\tan\phi=-h_x$) and $\phi_s$ is the internal angle of friction of the granular materials. $\Theta_{c0}$ is the critical value of the Shields parameter to set grains into motion on a flat horizontal sand bed. Note that q is proportional to a volumetric flux: $q_b = c\,[(s-1)gd^3]^{1/2}$ (c is a numerical constant).

Finally, the model is closed by the mass conservation equation for the grains:

$$\partial_t h = \partial_x q \qquad (3)$$

The trivial stationary solution of the model equations corresponds to a simple linear shear flow over a flat horizontal sand bed. The flow profile and the sediment flux are given by:

$$u_0(z) = U_0\; z/L = \gamma\, z$$
$$q_0 = q_b\left(\Theta_0 - \Theta_{c_0}\right) \qquad (4)$$

where $\Theta_0=\nu\gamma/g(s-1)d$ is the Shields parameter characterizing the flow over a flat bed.

We shall first recall the result of the linear stability analysis of the flat sand bed. The equations of the flow are usually solved using the quasi-stationary approximation, since the typical hydrodynamical time is generally much smaller than the typical morphological time of the bed. Within this approximation, the dispersion relation for modes of the form $h(x,t) = h_1 \exp(i\,k\,x + \omega\,t)$ (k being the wave number and ω the growth rate) can be derived (Charru et al. 2002, Valance et al. 2005). In the deep water approximation and the long wavelength limit, one finds:

$$\Re(w) = \frac{3q_b}{2l_\nu}\Theta_{c_0}^{\;3/2}\left[\;Ai(0)\mu(1+\mu)k^{4/3}l_\nu^{\;4/3} - \frac{k^2 l_\nu^{\;2}}{\tan\phi}\right]$$
$$\Im(w) = \frac{3q_b}{2l_\nu}\Theta_{c_0}^{\;3/2}\left[\sqrt{3}\;Ai(0)\,k^{4/3}l_\nu^{\;4/3}\right] \qquad (5)$$

where Ai is the Airy function and $l_\nu=(\nu/\gamma)^{1/2}$ is a 'viscous' length. We have introduced the parameter $\mu = (\Theta_0-\Theta_{c0})/\Theta_{c0}$, which will be referred to as the relative shear stress excess. The real part of the growth rate consists of two different terms: the first one, which scales as $k^{4/3}$, plays a destabilizing role and results from fluid inertia; the second one, proportional to $k^2$, stabilizes the sand bed at larger wavelength and is due to the slope effect in the sand transport law. As a result, there exists a band of unstable modes. The wave number and the growth rate of the fastest growing mode are given by:

$$k_{\max} = (\tan\phi_s)^{3/2}\,\frac{(1+\mu)^2}{5 l_{\nu_0}}$$
$$\omega_{\max} = \frac{\sqrt{(s-1)gd^3}}{{l_{\nu_0}}^2}\,{\Theta_{c_O}}^{3/2}(\tan\phi_s)^2\,\mu^{1/2}(1+\mu)^4 \tag{6}$$

where we have introduced the length $l_{\nu 0} = \nu/[(s-1)\,gd\Theta_{c0}]^{1/2}$, which corresponds to the critical value of the viscous length $l_\nu$ at the onset of grain motion. The sand bed is unstable as soon as the shear rate is strong enough to set the grains into motion (i.e., $\mu > 0$). The most dangerous mode is expected to prevail in the first stages of the instability and should give an order of magnitude of the initial wavelength of the ripples. This linear analysis predicts that the latter decreases with increasing shear stress $\mu$ and increasing internal friction angle $\phi_s$.

## 3 NONLINEAR ANALYSIS

To investigate the subsequent evolution of the ripples, it is necessary to go beyond the linear stability analysis and take into account the nonlinearities. We therefore performed a weakly nonlinear analysis via a multiple scale scheme. First we should introduce an appropriate small quantity $\varepsilon$, that we choose to be equal to $\tan\phi_s$. For standard internal angles of friction (from $20^0$ to $30^0$), $\tan\phi_s$ is comprised between 0.3 and 0.5. We will assume however that $\tan\phi_s$ is sufficiently small (typically of order of 0.1) and will extrapolate the results of our analysis to greater values of $\tan\phi_s$. The wave number of the fastest growing mode and its growth rate can be expressed in terms of $\varepsilon$. One finds that $k_{max} \sim \varepsilon^{3/2}$ and $\omega_{max} \sim \varepsilon^{2}$. A long-wave equation should be therefore derivable. In a multi-scale analysis, we introduce slow spatial variables $X = \varepsilon^{3/2} x$, $Z = \varepsilon^{3/2} z$, and a slow time variable $T = \varepsilon^2 t$. The strategy is then to rewrite the system equations in terms of the new variables and to make an expansion in power of $\varepsilon$. The system equations are solved at successive order and the sought non-linear equation for the bed profile arises as a compatibility condition. This multi-scale analysis is quite standard and details can be found in (Kassner et al. 2002).

In the sequel, we will use dimensionless variables: lengths will be reduced by $l_{\nu 0} = \nu/[(s-1)\,gd\Theta_{c0}]^{1/2}$ and time by $\tau = {l_{\nu 0}}^2/[(s-1)gd^3]^{1/2}$. The nonlinear analysis yields to leading order:

$$h_T = -A\,\partial_X\left\{ B\left[\aleph_{1/3}(h) - h_X\right] + D\left[\aleph_{1/3}(h) - h_X\right]^2 + Ch\,\aleph_{2/3}(h) \right\} \tag{7}$$

where

$$\aleph_q(h) = \Gamma(q)\sin(q\pi)\int_{-\infty}^{X} dX'\,\frac{h_X(X')}{(X-X')^q} \tag{8}$$

A,B,C and D are constant parameters and read:

$$\begin{aligned} A &= (3/2)\,{\Theta_{c_0}}^{3/2}\mu^{1/2} \\ B &= (3/2)\pi\,Ai(0)\,(1+\mu)^{4/3} \\ C &= 3\pi\,Ai'(0)\,(1+\mu)^{5/3} \\ D &= \Theta_{c_0}/4 \end{aligned} \tag{9}$$

$\aleph_q(h)$ is a Hilbert-type integral and its Fourier transform reads simply $2|K|^q\,\hat{h}\exp(iq\pi|K|/2K)$ (for $0<q<1$). In eq. (7), we have neglected high order nonlinear contributions in h (i.e., terms scaling as $h^n$ with $n>2$).

Upon rescaling of space, time and amplitude ($X \to X/B^{3/2}$, $h \to h/B^{3/2}$ and $T \to T/AB^3$), the above equation can be rewritten in a form in which only two parameters survive. Therefore, all parameters can be set to unity except two, denoted below as a and b:

$$h_T = -\partial_X\left\{ \left[\aleph_{1/3}(h) - h_X\right] + a\left[\aleph_{1/3}(h) - h_X\right]^2 + bh\,\aleph_{2/3}(h) \right\} \tag{10}$$

a and b can easily determined: $a = D = {\Theta_{c0}}^{3/2}/4$ and $b = C/B^2 \approx 4.3/(1+\mu)$. This nonlinear equation is non-local. The non-locality appears in the linear terms as well as in the nonlinear ones and is due to the long-range interactions mediated by the fluid. To our knowledge, it is the first time that such a nonlinear and non-local equation has been derived in the context of sand bedforms. Similar non-local equations but with different types of nonlinearities have been however reported in other contexts such as in the case of the Grinfeld instability for elastic strained solids (Kassner et al. 2002).

## 4 RESULTS

Let us now investigate the features of eq. (10). First, it can be easily checked that the linearization of this equation produces the linear dispersion relations [ eq. (5)], which reads in its dimensionless form as

$\omega=K^{4/3}-K^2-i\sqrt{3}\,K^{4/3}$. Second, the numerical resolution of eq. (10) shows that an initially random rough bed evolves towards a ripple pattern, which exhibits at long time a coarsening process (i.e., the wavelength increases in course of time) (see Fig. 2).

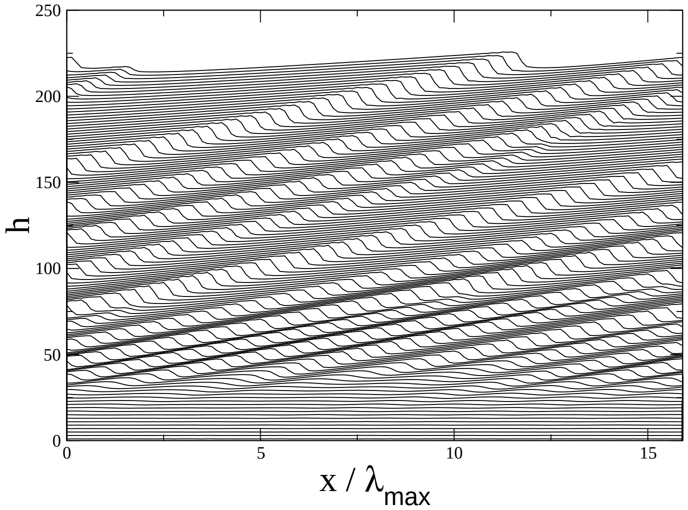

Fig. 2: Spatio-temporal diagram showing the evolution of an initial random rough sand bed. a=0.1 and b=0.

For the parameters investigated so far (i.e., $0<a<1$ and $0<b<1$), the coarsening process never stops: the mean wavelength grows indefinitely and no final state is selected. Fig. 3 shows a typical evolution of the mean wavelength and amplitude of the ripple pattern as a function of time starting from an initial random rough sand bed. One can note that before the coarsening process operates, there is a transient regime where the mean ripple wavelength remains constant (and is equal to the linearly most unstable mode) while its amplitude grows exponentially fast. When coarsening proceeds, the wavelength and amplitude both exhibit a power law behavior (i.e., $\lambda\sim t^{\xi}$ and $A\sim t^{\chi}$). The scaling exponents are not much sensitive to the equation parameters a and b. One finds $\xi\approx0.81\pm0.02$ and $\chi\approx0.27\pm0.02$. In addition, the drift speed of the ripple pattern is also found to obey a power law, $v_d\sim t^{\psi}$ with $\psi\approx0.22\pm0.02$.

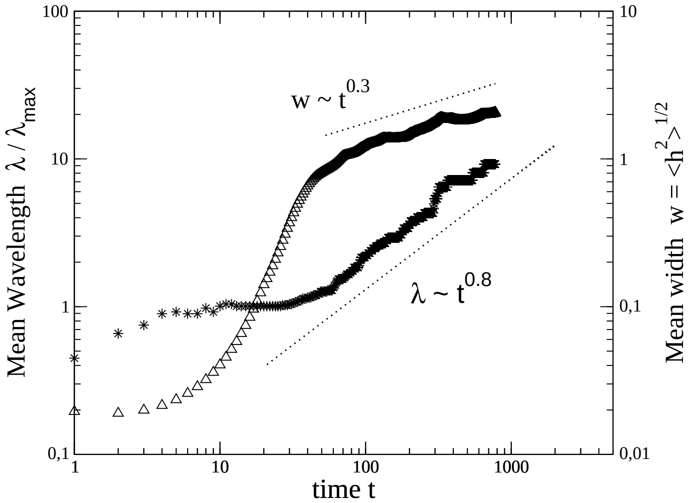

Fig. 3: Evolution of the mean wavelength and mean width of the ripple pattern in course of time. The initial bed profile is random and the system size L is equal to 64 $\lambda_{max}$. a=0.1 and b=0.

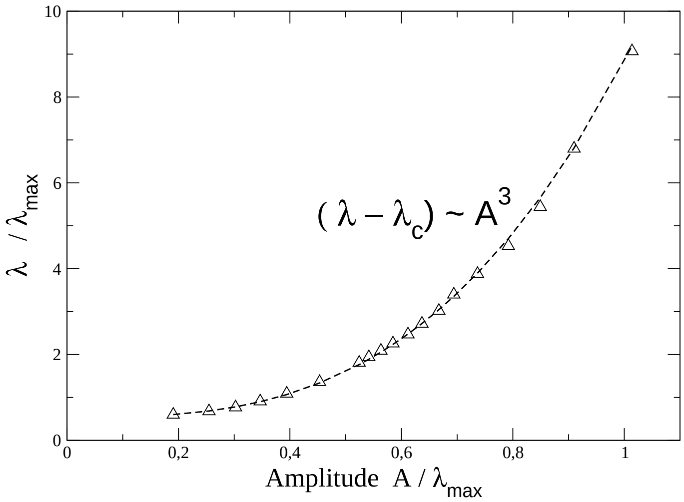

Fig. 4: Amplitude A of the steady-state solutions versus periodicity λ.

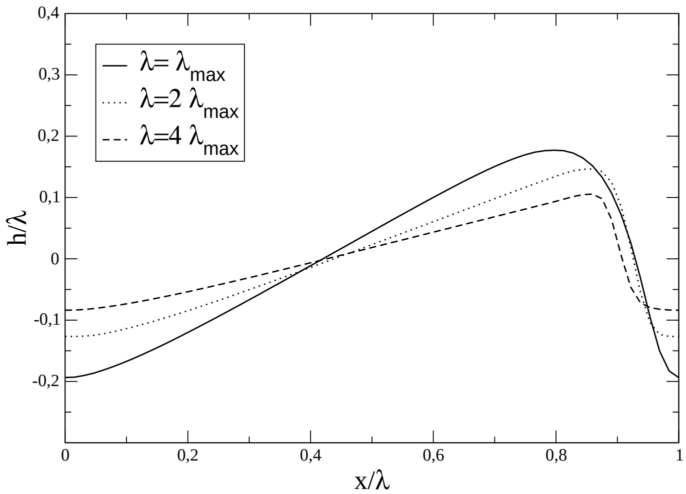

Fig. 5: Profile of the steady-state solutions for different periods. a=0.1 and b=0.

It is instructive to characterize more precisely the morphology of the ripple pattern in course of the coarsening process. From the previous results, one deduces that the amplitude A increases with increasing $\lambda$ while the ratio $A/\lambda$ decreases with increasing $\lambda$ (since $\chi<\xi$). The ripples become therefore more and more elongated as coarsening proceeds: there is no scaling invariance. A simple way to check these results is to determine the steady state solutions of eq. (10) with a given periodicity $\lambda$. Fig. 4 displays the amplitude of the steady solutions as a function of the periodicity $\lambda$ and their corresponding profiles. The data have been obtained for given values of the equation parameters (i.e., a=0.1 and b=0) but the features remain qualitatively unchanged for the range of values investigated so far. One finds in particular that the ripple amplitude A scales as $(\lambda-\lambda_c)^{1/3}$ (where $\lambda_c$ is the cut-off wavelength below which all the mode are stable; $\lambda_c=2\pi$). One can note that the ratio $\chi/\xi$ is not far from 1/3. This strongly suggests that the analysis of steady-state solutions provides reliable information with respect to the ripple dynamics and supports the statement established recently by Politi and Misbah (Politi et al. 2004) that the coarsening process of one-dimensional fronts occurs only if the periodicity $\lambda$ of the steady-state solutions is an increasing function of its amplitude A.

We also determined the drift speed $v_d$ of the stationary patterns and found that $v_d \sim \lambda^{1/3}$ for $\lambda > \lambda_c$. It follows that for large wavelengths (i.e., $\lambda<<\lambda_c$), the migration speed of the ripple is inversely proportional to the ripple amplitude, $v_d \sim 1/A$. This result is similar to that found for barchane dunes in the context aeolian sediment transport (Andreotti et al. 2002).

## 5 CONCLUSION

In conclusion, we derived a non-local and nonlinear equation for the dynamics of sand ripples sheared by a two-dimensional liquid flow of infinite depth. The resolution of this equation shows that coarsening occurs and no final state is selected. Within this flow configuration, there is no mechanism able to interrupt the coarsening process. This result is supported by experimental findings of Loiseleux et al. in a quasi two-dimensional flow of large water depth (Loiseleux et al. 2004). The three dimensionality of the flow may cause an inhibition of the coarsening. Indeed, it has been shown recently that unstable transverse modes can couple to longitudinal ones in a nonlinear way and gives birth to steady two-dimensional patterns (Langlois et al. 2005). This may explain why coarsening stops in the experiment of Bass (Bass 1994) where the flume is wide enough in order that transverse modes can develop. Moreover, the shallowness of the flow may also cause the interruption of the coarsening. The experiments of Rehberg et al. (Rehberg et al. 2002), in which the amplitude of the final ripple pattern is of order of the flow height, seem to confirm this hypothesis. It would be therefore interesting for the future to test the above hypotheses by solving numerically in 2D and 3D the full Navier-Stokes equations coupled to the sand transport law.

## 6 REFERENCES


Andersen, K.H. & Abel, M. & Krug, J. & Ellegaard, C. & Sondergaard, L.R. & Udesen, J. 2002. *Phys. Rev. Lett.* 88, 234302.

Andreotti, B. & Claudin , P. & Douady, S., 2002. *Eur. Phys. J. B 28*, 321.

Baas, J.H. 1994. *Sedimentology 41*, 185-209.

Betat, A & Kruelle, C.A. & Frette, V. & Rehberg I. 2002. Eur. Phys. J. E 8, 465.

Charru, F & Mouilleron-Arnould, H.. 2002, *J. Fluid Mech*, 452, 303.

Csahok, Z. & Misbah, C. & Rioual, F. & Valance, A. 2000. *Eur. Phys. J* E 3, 71.

Fredsoe, J. & Deigaard, R. 1992. Mechanics of coastal sediment transport , World Scientific.

Hansen, J.L. & vanHecke, M. & Ellegaard, C. & Andersen, K.H. & Bohr, T. & Sams, T. 2001, *Phys. Rev. Lett.* 87, 204301.

Langlois, V. & Valance, A. 2005. *Phys. Rev. Lett.* 94, 248001.

Kassner, K. & Misbah, C., 2002. *Phys. Rev.* E 66, 026102.

Loiseleux, T. & Doppler, D. & Gondret, P. & Rabaud, J.-M. 2004. *Second International Workshop on Marine Sandwave and River Dynamics* (edited by J.M. Hulsche, Y. Garland, D. Idier, University of Twente).

Nishimori, H. & Ouchi, N. 1993. *Phys. Rev. Lett.* 71, 197.

Politi, P. & Misbah C. 2004. *Phys. Rev. Lett.* 92, 090601.

Valance A. & Langlois, V. 2005. Eur. Phys. J. B 43, 283.

Werner, B.T. & Gillpsie, D.T. 1993. *Phys. Rev. Lett.* 71, 3230.