A simplified sand wave model

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Abstract
Based on the Swift-Hohenberg equation [1] an efficient method for predicting the behaviour of sand waves is constructed. The goal is to investigate whether random small disturbances in the initial conditions can account for spatial and temporal variations in sand waves. Existing models cannot be used to do calculations on the variations in space and time of sand waves, either because they are too complicated to use in actual calculations, or because they lack the necessary sophistication to describe what happens. The new method is used to show that random disturbances in the initial conditions can be a cause of the spatial and temporal variations.

1. Introduction
Sand waves form a pattern of more or less parallel ridges. The wave length is about 300 meters and the height up to 10 meters, which is a considerable amount of the total water depth. Sand waves migrate with speeds of about 10 meters per year. Information on their behaviour is valuable: the larger part of the south of the North Sea bottom is covered with sand waves and there are plans to use this region for sand extraction, the placement of windmills and even the construction of an artificial island. Knowledge of the implications of such plans is needed to make decisions about them.

The currently used models that describe sand wave behaviour, consist of two parts. First the Navier Stokes equations are used to calculate the tidal flow. Next a sediment transport model describes how a bottom subject to that tidal flow develops. The tidal flow needs to be calculated on a small time scale and the sediment transport on a long time scale. Together, the system is complicated and it takes large amounts of time to calculate. For this reason, only the effect of sine formed disturbances has been investigated (linear stability analysis [2] and [4]). This way the occurrence of sand waves can be predicted. The predicted waves are all equal in size and shape and there is no (non-trivial) dynamic behaviour. In reality, however, the amplitudes and wave lengths of natural sand waves do vary in space and time (fig.1). The reasons for this are not yet theoretically understood.

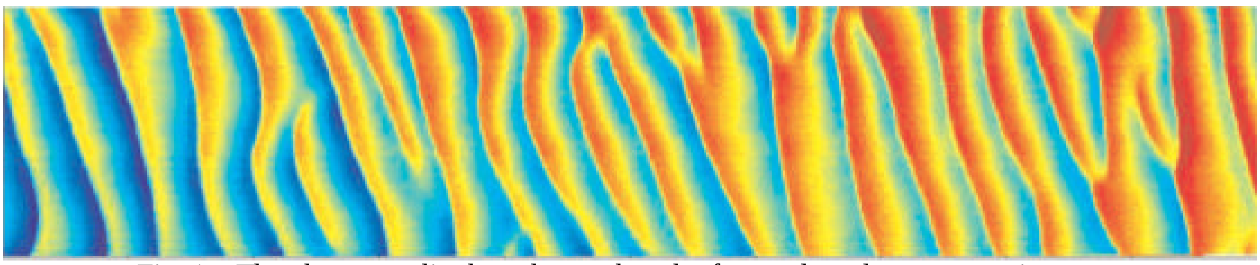

Fig. 1 – The phase, amplitude and wave length of natural sand waves vary in space.

In this article we look at a more efficient technique to predict the development of sand waves. The goal is to find a model combined with a calculation method that is applicable to arbitrary initial conditions. This in order to be able to investigate whether irregularities in the initial conditions could be the cause of variations in sand wave patterns found in nature. An added benefit of such a model is that it can be used to predict the effect of large real life disturbances caused by human interference, like e.g. dredged channels, dams and artificial islands. These disturbances are usually not harmonic.

In the next section a possible model is derived. In section 3 an efficient way to perform calculations with that model is described. In section 4 the technique is extended to 2D. The extended model is used to investigate whether randomness of the initial disturbances can be a cause of variations in space and time in the developing sand waves. The results are given in section 5. In the last section conclusions are given.

2. Model equation

Instead of directly using the physical model to calculate how sand waves evolve, we examine the characteristics of the system of equations. The following observations can be made:

- The linear growth is a good indicator for the wave length of occurring sand waves. Applying linear stability analysis on the complete physical model gives the linear growth rate for a given wave length (Hulscher [4] and Gerkema [2]). Research by Hulscher and Van den Brink [3] has shown that the fastest growing mode matches the wave length that occurs in real life.
- Linear growth in combination with a non linear term gives a good description of the final amplitude behaviour. Knaapen [5] has used a Ginzburg-Landau based equation to predict the regeneration of sand waves after dredging. Although the model only incorporated wave length independent growth and a very basic non linear term to limit growth, good prediction quality was already achieved. There is a limitation to this approach: it can only be used if the dredging has not changed the original wave lengths. If not just the tops of the waves, but the complete structure is dredged away, this model fails.

The desired model should have correct, wave length dependent, linear growth and a simple non linear term to limit growth. The most simple non linear term that provides damping of the amplitude and does not alter wave lengths is h^3. We get the following equation for the bottom disturbance h:

$$\partial_t h + \mathcal{L}(h) + a h^3 = 0 \tag{1}$$

where $\mathcal{L}$ denotes a linear differential operator that gives the desired wave length dependent growth. We choose:

$$\mathcal{L}(h) = l_1 \partial_x^2 h + l_2 \partial_x^4 h \tag{2}$$

This operator has a wave length dependent growth w:

$$\omega = l_1 k^2 - l_2 k^4 \tag{3}$$

The wave number k is related to the wave length $L=(2p)/k$. The parameters l_1 and l_2 can be chosen such that the growth curve (see fig. 2) fits the linear growth found by Hulscher [4].

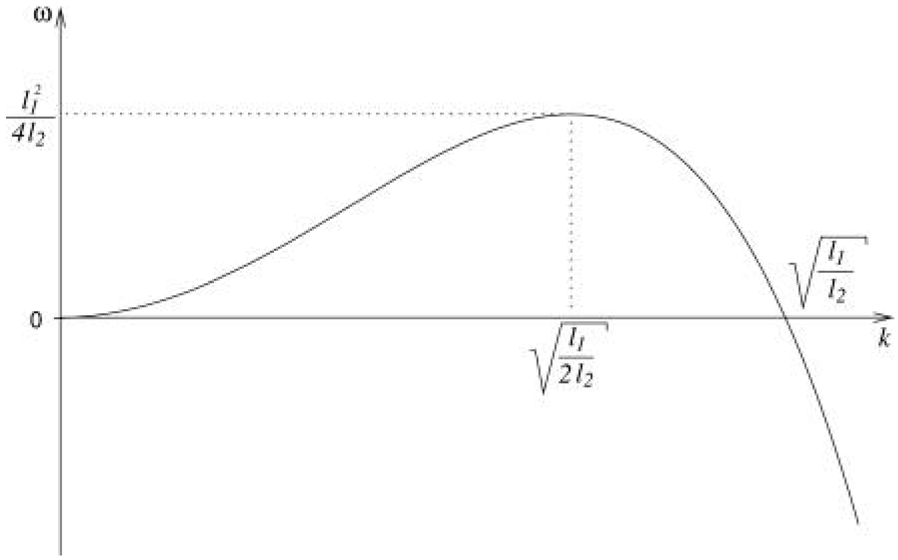

Fig. 2 – Linear growth ω (equation (3)) against the wave number k. Short waves (k high) are dampened out completely, very long waves remain what they are and the ones in between are amplified.

The coefficient *a* determines the final amplitude of the sand-wave. By substituting

$$h = \sum_n a_n \sin(nkx) \tag{4}$$

and $\partial_t h = 0$ (stationary) into equation (1) and using only the first harmonic we get:

$$\left[a_1(l_2k^4 - l_1k^2) + \frac{3}{4}aa_1^3\right]\sin(kx) - \frac{1}{4}aa_1^3\sin(3kx) = 0 \tag{5}$$

Again truncating at the first harmonic and solving for a1 gives the final amplitude a1 for each value of k:

$$a_1 = \frac{2k\sqrt{l_1 - l_2k^2}}{\sqrt{3a}} \tag{6}$$

3. Solving method

In order to calculate the development of sand waves using equation (1), an efficient way of solving is needed. Because the equation by itself is linear unstable, it is not trivial to find a numerical scheme that is stable and accurate.

The purpose of the linear operator $\mathcal{L}(h)$ (eq. (2)) is to give wave length dependent linear growth. An elegant way to apply this growth is perform the calculations in the Fourier domain. Using a fast Fourier transform, any discrete bottom state can be decomposed into a series of sine and cosine waves and back, without loss of accuracy. In other words, in the Fourier domain the bottom state is split up into a combination of wave lengths, each of which can be subjected to its corresponding linear growth. If we multiply the Fourier transform with a suitable amplification function *A(k)*, the result of $\mathcal{L}(h)$ over a certain time step *Dt* can be calculated directly. The amplification function is the exponential growth during the time step *Dt*.

$$A(k) = e^{\omega(k)\Delta t} \tag{7}$$

This technique is unconditionally stable and exact.

The non linear part can not be calculated efficiently in the Fourier domain. It is applied in the real domain using a fourth order Runge Kutta time integration.

4. Extension to 2D

In order to be able to represent phenomena like forks in the sand wave pattern, two horizontal directions are needed. In order to do the 2D calculation correctly a new linear operator $\mathcal{L}$ can be introduced:

$$\mathcal{L}(h) = l_1\partial_x^2 h + l_2\partial_x^4 h - l_3\partial_y^2 h \tag{8}$$

Like the parameters l_1 and l_2, the new parameter l_3 can be chosen such that the growth curve matches the one found by Hulscher [4].

Because we apply the linear growth in the Fourier domain, a more sophisticated linear operator can be used. In stead of using a $\mathcal{L}(h)$ fitted to the linear growth found by Hulscher [4], it is possible to use the actual linear growth curve. The continuous growth curve can be evaluated at the discrete Fourier points. With these discrete values for w the correct amplification function $A(k)$ (7) can be determined.

5. Application

In linear analysis the growth of a harmonic disturbance is investigated. In real life the small disturbances will not be harmonic, but random. In order to find out whether this randomness could be the cause of the irregularities found in sand wave patterns, we apply the method described in the previous sections to a flat bottom with small random disturbances. First this is done for a 1D topology and then for a 2D one.

In figure 3 the results for a 1D test case with random initial data are given. In the random initial condition all wave lengths are present. In the start of development the short wave lengths are dampened out completely. Locally waves start to grow independently. If two wave regions with the optimal wave length and a different phase meet, a transition between the two is needed. Most of the time the wave length of the transition will not be optimal, resulting in a smaller amplitude. This is in correspondence with equation (6). After a while the sub optimality will be distributed into both wave regions. At the end all regions will be shifted such that the complete field has the optimal wave length.

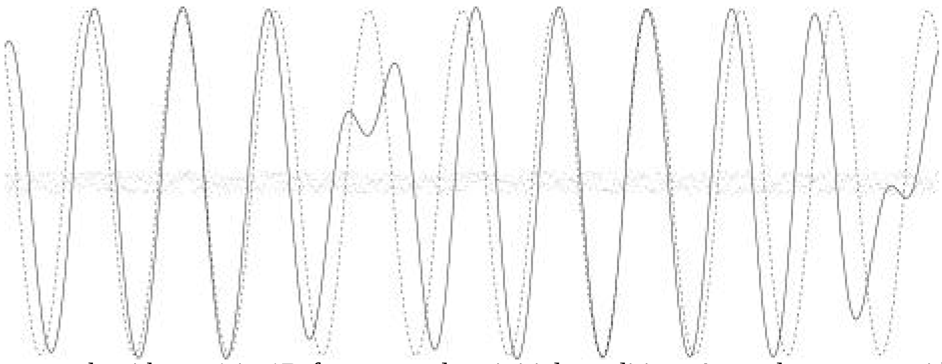

Fig. 3 - Finite growth with eq. 1 in 1D from a random initial condition. A regular pattern with the optimal wave length is eventually formed. For non-optimal wave lengths the amplitude is smaller.

In the 1D test case variations in wave length and amplitude can already be found from random initial data. Phenomena like forks in the wave pattern are however inherently 2D, and can not be found. In order to be able to look at forks and variations in sand wave bar length, the same approach is tested in 2D. The results can be found in figure 4. Solving with a random initial condition in 2D gives realistic patterns with different wave lengths, amplitudes and bar lengths.

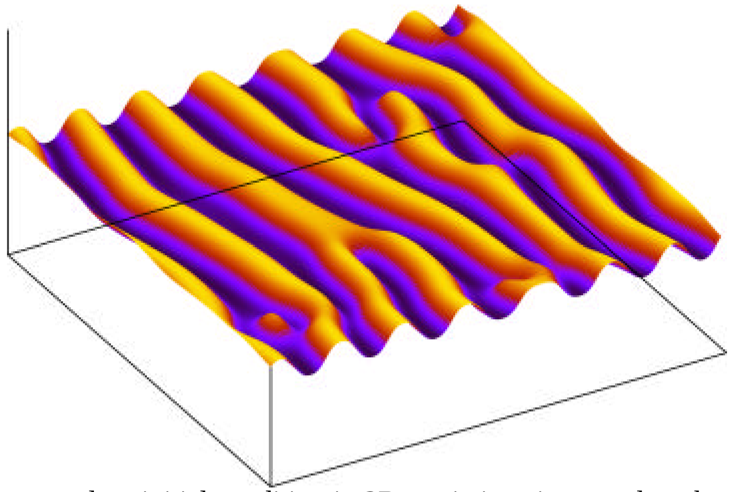

Fig. 4 - Starting from a random initial condition in 2D, variations in wave length and amplitude can be found, as well as forks.

After a long time (corresponding to about 7000 years, depending on the size of the domain), the end pattern will eventually become completely regular.

6. Conclusions

The newly derived method can be used to do calculations on large domains in two horizontal directions very efficiently. Applying it to a random initial condition results in patterns that much resemble the ones found in nature. This is an indication that the variations in space and time of sand waves are (partially) caused by the irregularities in the initial condition. There will however probably be other causes too, because the completely regular end patterns that the model predicts (as a final solution), are not found in reality.

Furthermore, the new method can be used to predict the effect of arbitrary disturbances caused by human interference e.g.: dredged channels, dams and artifical islands.

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