Modeling bedform evolution as an interface problem : Comparing bedform kinematics generated by advection-diffusion equations against reality

D. Jerolmack* and D. Mohrig

Department of Earth, Atmospheric and Planetary Sciences, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Building 54-820, Cambridge, MA 02139, USA
*douglasj@mit.edu; fax: 617-253-6385

Abstract
Bedform evolution remains dynamic even in the special case of steady, uniform flow. We describe such interactions between bedforms with data collected in the laboratory and a river. The kinematic data is used to motivate development of linear and nonlinear advection-diffusion models depicting development of topography at the sediment/fluid interface. Many qualitative features of bedform behavior are observed, but important properties such as the dispersion that controls bedform splitting and merging are not accurately reproduced. Limitations of the one equation models point to the necessity of including a second equation that explicitly treats sediment transport. The BCRE equations, originally developed for grain avalanches, are presented as a viable model for investigating the transient behavior of bedforms.

1. Motivation and philosophy
Observations collected in natural systems and laboratories demonstrate that bedforms are capable of generating their own internal dynamics due to the complex feedback between the flow and sediment-transport fields and the evolving surface topography. This dynamic manifests itself as a natural variability in bedform height, length, and celerity, merging and splitting, even when the topography is developing under what would otherwise be steady and uniform flow conditions. These internal adjustments are well known qualitatively (e.g., **Allen 1973**) but have received relatively little quantitative treatment by the scientific community. Instead, a great deal of attention has been given to development of algorithms relating mean flow and transport properties to mean bed topography (e.g., **Yalin, 1964; Fredsoe, 1982; van Rijn, 1984**). Such models serve as useful 'rules of thumb' describing average bedform properties, particularly height, length and celerity, but cannot be used to explore the interactions between bedforms that in fact determine their mean properties. An incomplete understanding of how irregular bed topography controls turbulence production and how this turbulence affects the local sediment transport precludes development of a bedform evolution model from first principles. While this result may seem discouraging, it has motivated us to develop new types of models aimed at reproducing the interacting topography. Bedforms are ubiquitous, occurring in subaerial, fluvial marine and submarine environments, and generated under a wide spectrum of flow and sediment-transporting conditions. It is clear a train of bedforms is a fundamental instability of the interface between fluid and sediment, and may therefore be only weakly dependent on the details of a particular system. One may wonder then if it is possible to treat the evolution of the fluid/sediment interface as a function of its shape alone by adapting generic interface equations.

Our analyses of topographic data capturing bedform evolution in time and in space reveal an extreme sensitivity to boundary conditions. In other words, the translation and deformation experienced by any given bedform strongly depends on the local configuration of the surface topography or interface. These data are derived from a 0.2m wide laboratory flume (**Jerolmack and Mohrig, submitted to J. of Geophys. Res., hereafter referred to as Jerolmack and Mohrig, in review**) and the North Loup River, Nebraska, USA (**Mohrig, 1994; Mohrig and Smith, 1996**). In both cases the subaqueous bedforms are composed of medium sand. Time series of topography from each environment are presented in Figure 1. Transient behavior of topography developing from two original bedforms is shown in Figure 1a. These stacked profiles record intervals of bedform translation, punctuated by deformation cascades during which bedforms undergo rapid splitting with highly variable celerity. All of this behavior is associated with a steady fluid discharge. The differences in bedform activities correspond to relatively subtle changes in interface geometry. In particular, minor changes in configurations of bedform troughs were found to increase local turbulence production by as much as five times (**Jerolmack and Mohrig, in review**). These differences in turbulence affected the sediment transport and subsequent bedform topography. Sequential profiles from the river bottom shown in Figure 1b record variability in bedform behavior comparable to that observed in the laboratory. In this time series the bed elevation scales with grayscale intensity. Crests and troughs of bedforms are relatively bright and dark, respectively. Tracing crest lines through time and space provides a quantitative measure of celerity. Bifurcations and terminations of crest lines represent the splitting and merging of individual bedforms, respectively. This 30m transect of river bottom has three distinct zones. A zone of high bedform activity from 8m to 15m at t = 0min separates two sections of lesser activities. In these two sections bedforms migrate with a mean celerity of 3cm/min and there is little change in their number through time. In the intervening zone the bedforms migrate with a mean celerity of 7cm/min and there are many bifurcations and terminations. Capturing these styles of variable behavior, as well as the discrete interactions between adjacent bedforms, is the motivation for our considering the evolving topography as an interface problem.

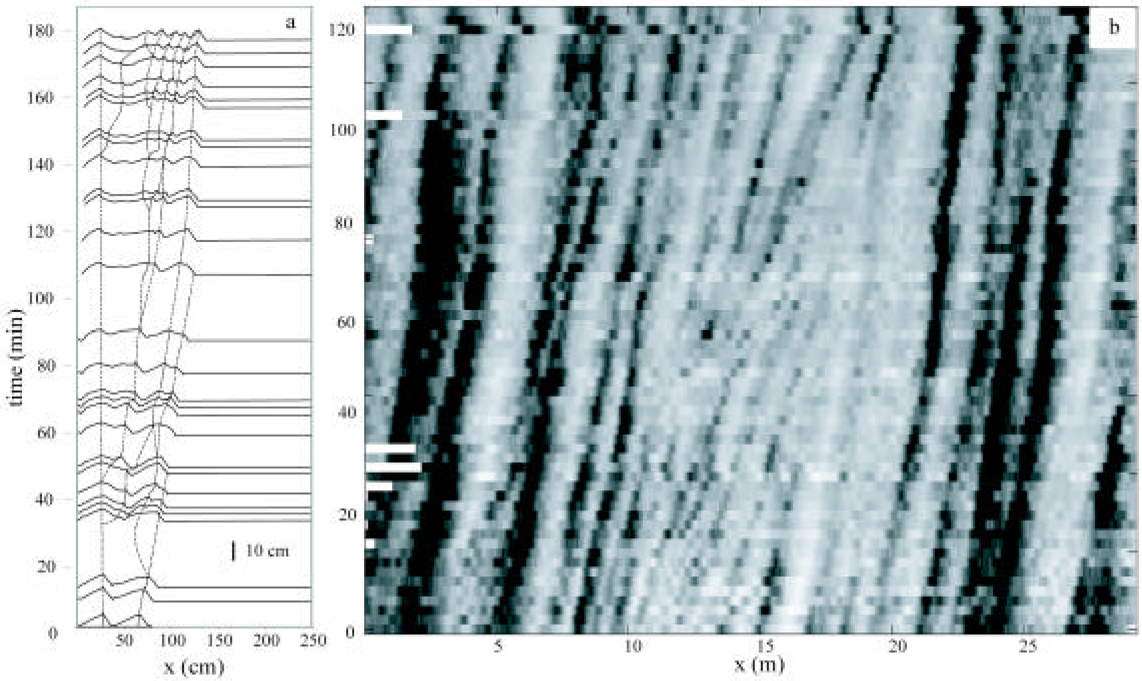

Figure 1. (a) Evolution of two laboratory dunes shown in profile. Dashed lines connect the same bedform crest on successive profiles. Beginnings and endings of dashed-lines mark the splitting and merging of individual bedforms, respectively. (b) Successive profiles for a train of bedforms in the North Loup River, NE, USA. Each profile is mapped in grayscale with brightness increasing as a function of elevation. Darkest and lightest coloring is correlated with the troughs and the crests of bedforms, respectively. Migration rates of individual bedforms are recorded as the slopes to the lines connecting their troughs and/or crests through space and time. Horizontal white bars mark missing data.

2. Bedform kinematics as an evolving interface

2.1. Describing bedforms using an advection-diffusion equation

We now proceed with a heuristic development for a model capturing the essence of bedform behavior. In order for this model to be considered successful it must at the very least provide meaningful descriptions of the following properties: 1) cross-sectional geometries for trains of bedforms; 2) splitting of unstable topographic elements into two or more bedforms; 3) merging of elements into single bedforms; and 4) natural variability in rates of bedform migration. We begin here by exploring the capability of an advection-diffusion equation to capture these necessary behaviors. A significant advantage of describing evolving topography with such an equation is that the shape of the bed at any given time *completely determines* the shape of the bed at the next time step. In this framework the evolutions of individual bedforms are specifically functions of bed slope, curvature and elevation. Characteristics of the flow and sediment transport acting to *transmit* information from one point to another are reduced to a small, interpretable set of coefficients and exponents.

Exner (1925) presented the first relevant description for the advective component of bedform evolution. Based on continuity arguments he demonstrated that the translation and deformation of a topographic perturbation can be described by

$$\frac{\partial z}{\partial t} = \Lambda\left(1 + B\frac{z}{h}\right)^n \frac{\partial z}{\partial x} \qquad (1)$$

where z is elevation, x is horizontal distance, $\partial z/\partial x$ is the interface slope, t is time, Λ is celerity or advective velocity of the interface [L T^{-1}], h is flow depth, B is a coefficient, and n is an exponent that can vary between 1/2 and 2 depending on the chosen transport relation. Equation 1 describes an elevation-dependent advection of topography. With this equation an original bump with Gaussian-distributed elevation becomes skewed in the direction of transport. The resulting asymmetry increases through time producing an ever lengthening overhang to the lee face of the topographic element. This unrealistic development can be mitigated by adding a diffusive term to the interface equation. This new term acts to damp topography by reducing curvature. Its inclusion is supported by observation of active bedforms in the lab and field documenting a diffusive component to their behaviors, especially when merging. The most general advection-diffusion equation (ADE) for the interface is then

$$\frac{\partial z}{\partial t} = \Phi\nabla^2 z + \Lambda\frac{\partial z}{\partial x} + h(z,t) \qquad (2)$$

where Φ is diffusivity [L^2 T^{-1}], $\nabla^2 z$ is curvature [L^{-1}], and $h(z, t)$ is a topographic source term. Here we have calculated curvature in either one (1D) or two (2D) dimensions, depending on the domain, while always restricting advection to the specified streamwise direction. Equation 2 is dissipative for $\Phi > 0$ and cannot grow topography from an originally flat surface when $\langle h(z, t)\rangle = 0$. To explore the development of topography described by variants of this equation we seed the solution matrix with an initial topography and monitor its transient evolution via numerical integration of (2). Resulting examples of 1D and 2D topography are presented in Figure 2.

The value for Λ in Equation 2 can be either a constant, yielding the linear ADE, or a function of topography. Making Λ an explicit function of elevation, where $\Lambda = \lambda z$ and λ is a constant [T^{-1}], yields the well-studied nonlinear ADE typically called the Burgers equation. This equation was originally proposed to describe the velocity field for simplified 1D turbulence (**Burgers, 1974**) and has subsequently been used to model interface growth (**Medina et al., 1989**), traffic (**Nagel 1996**) and other systems. We use Burgers equation as our model nonlinear ADE, noting that it is essentially Exner's (1925) expression for bedform evolution (1) with a diffusive term added to it.

2.2. Linear advection-diffusion equation

Equation 2 with Λ = constant provides a description of transient bed evolution that does not capture any of the bedform activity we are interested in studying. In particular this expression generates none of the interactions between topographic elements that can be interpreted in the context of bedform splitting and merging. Seeding a matrix with high-amplitude random topography and monitoring its development by numerically integrating (2) produces a series where every point in the bed profile translates downstream with a velocity Λ while the curvature to topography is systematically reduced at a rate controlled by Φ. An example of this style of bed evolution is shown in Figure 2b. While not useful for exploring bedform interactions, we found the linear ADE to be helpful tool when analyzing topographic data sets. We calculated slope and curvature for all 66 bedform profiles shown in Figure 1b. Successive profiles were then differenced to find elevation change at each point as a function of time and local values for Λ and Φ were back calculated using (2). Resulting distributions of each parameter yielded a central value for Φ that was between 1 and 2 orders of magnitude smaller than the central value for Λ. With these values for Λ and Φ in hand a very simple set of forward calculations can be performed that begin to quantify rates of translation versus deformation in the system. This forward-looking description for bedform topography neglects Φ and uses the median value of Λ to forecast from one time step to the next. Differences between the predicted and measured profiles at that step allow us to quantify how much of the profile evolution is attributable to 'steady-state' translation versus irrecoverable deformation.

2.3. Burgers equation

Equation 2 with $\Lambda = \lambda z$ provides a description of transient bed evolution possessing many aspects of bedform activity we are interested in studying. Seeding a matrix with high-amplitude random topography and monitoring its development by numerically integrating (2) produces a series where the initial bumps migrate at different speeds depending on their height, and merge irreversibly due to diffusion. Within a short time, asymmetric 'bedforms' are generated with variable wavelengths. By additional mergers the bedforms become increasingly periodic with time and achieve a characteristic wavelength at long time. In 2D these features are sinuous-crested and form coherent structures that may occupy the entire cross-stream width of the domain (Fig. 2a). Since (2) is dissipative the bed eventually decays to a flat surface if no additional forcing is added to the system.

We can now add a low-amplitude white noise, $\langle\eta(z, t)\rangle = 0$, and examine bed evolution in the noisy case. The noisy Burgers equation has been explored by several researchers (e.g., **Fogedby, 1999**) who have found that the additional roughness associated with even very-low amplitude noise preserves transients over substantially greater time intervals. This is because diffusion acts on the local curvature and preferentially dissipates the noise topography, thus preserving larger waves. Interestingly, noise enhances wavelength variability even though its amplitude is much smaller than the resultant waves. Amplification of noise by the nonlinear advection term generates this increased variability in the system.

Figure 2b compares the profile evolution described by a linear ADE, Burgers equation and a noisy Burgers equation for an initial sinusoidal profile. In the case of Burgers equation each topographic element skews in the direction of transport and develops a smooth shock, forming a breakaway head that migrates away from the rest of the body. If diffusion is small, several generations of breakaways may occur before all elements reach such low amplitude and/or slope that further development of smooth shocks is inhibited. With addition of noise, we see the same general pattern; however there are bedforms of different wavelengths on the "backs" of larger bedforms, and evolution is much more variable. This noisy Burgers case bears qualitative similarities to Figure 1a.

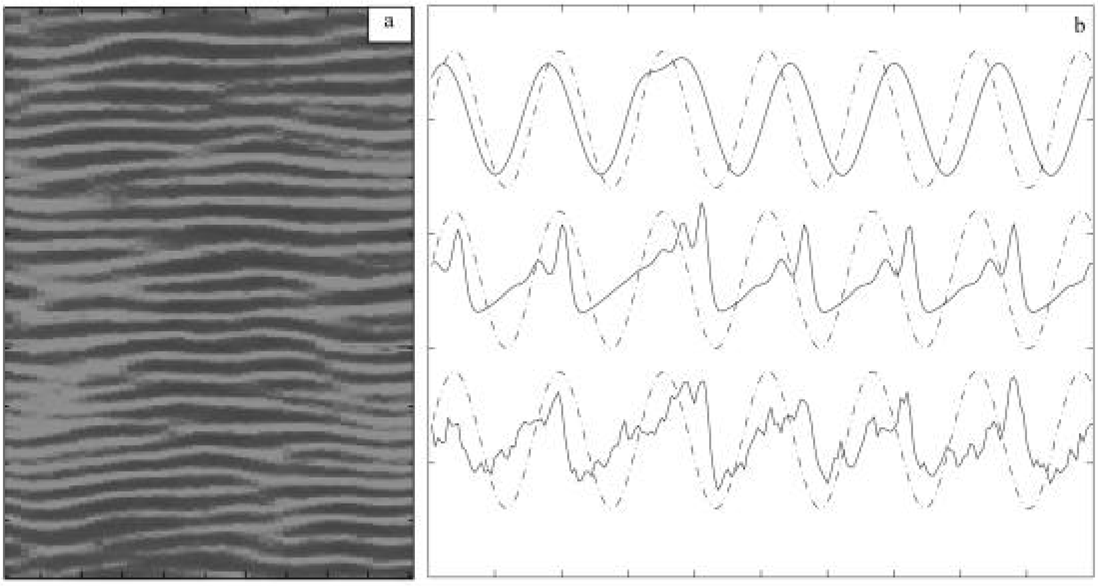

Figure 2. Bed topography generated by reported advection-diffusion equations. Equations integrated using a two-step Lax-Wendroff method. Boundary conditions are periodic in all plots. (a) Topographic map at t=1000 generated by 2D Burgers equation from random initial topography. Elevation increases with brightness. Transport is from top to bottom of domain; grid is 50×200. (b) 1D evolution of sinusoidal topography (dotted lines) with linear ADE (top), Burgers (middle) and noisy Burgers (bottom) at t=1000. Advection from left to right, domain length is 200. Anomalous bedform shape at t=1000 is due to a mismatch of the left and right domains at the initial condition.

3. What have we learned, and what are we missing?

We will first comment on insights gained from using the Burgers equation. Bedform-like topography 'self organizes' from random initial topography under all conditions and the process of merging appears realistic. Bedforms skew naturally in the direction of migration and can split as they become unstable. It is difficult to obtain a bedform field that focuses splitting in some regions and merging in others (Fig. 1b), but we guess that with a more sophisticated treatment of coefficients, i.e. diffusivity varying as some function of elevation rather than being constant, we may obtain even more interesting behavior from this single equation. Burgers equation produces topography with sinuous crest lines, bifurcations and defects (Fig. 2a). These strikingly realistic geometries are not specified *a priori*. Bedform spacing is generally quasi-periodic with some inherent variability. The introduction of a noise term enhances this variability. Another attractive feature of this simple model is that reversing (oscillatory) and combined flow transport can be treated easily and reasonable looking topography is generated. Many salient characteristics of bedforms are reproduced by treating the bed simply as an interface that evolves as a function of its elevation, slope and curvature.

We recognize that there are major deficiencies in the one-equation description of bed evolution. First off, the model equation is dissipative and bedforms are not self-sustaining. The model equation cannot *grow* topography, rendering it incapable of treating bedform initiation. Predicted dispersion, the relationship between celerity of waves and their size, is unrealistic. With a linear ADE there *is no dispersion relation*, all topography simply translates at the same speed. In Burgers equation points of higher elevation move faster, making celerity proportional to bedform height (*H*). This relationship is exactly *opposite* of anticipated relationship for natural bedforms (**Raudkivi and Witte, 1990**). While in natural systems $\Lambda \sim 1/H$ may not hold strictly (e.g., **Carling et al., 2000**), it is not encouraging that our dispersion relation is opposite of that observed in most laboratory experiments and predicted by the conservation of sediment mass. What are we missing?

The well-known dispersion relation, $\Lambda \sim 1/H$, is derived by integrating sediment flux over one entire bedform and assuming equilibrium topography. This leads us to suspect that we need to treat sediment transport explicitly to obtain the proper dispersion relation in our model. This may also allow us to build topography from scratch. The BCRE equations originally proposed by Bouchaud, Cates, Ravi Prakrash and Edwards (**1995**) to model grain avalanches have been recently adapted to study additional types of granular flows, including wind ripples and dunes (**Terzidis et al., 1998**). The model treats two species of grains; a static or immobile layer, which is our normal bed elevation, and a thin overlying layer of mobile grains with thickness $\Re$ [L]. Immobile grains can be converted to mobile grains, causing a local decrease in elevation, or mobile grains can be deposited and contribute to an increase in height. The BCRE equations are

$$\frac{\partial z}{\partial t} = \Re\left(\Phi\nabla^2 z + \Lambda\frac{\partial z}{\partial x}\right) \qquad (3)$$

$$\frac{\partial \Re}{\partial t} = \frac{\partial(\nu\Re)}{\partial x} + \frac{\partial z}{\partial t} \qquad (4)$$

where ν is advective velocity of the mobile grains [LT^{-1}]. The equations are coupled, and (3) is nonlinear. Stability of these equations has been treated extensively in the physics literature, but the range of behavior they produce has not been exploited, nor have they been specifically adapted for subaqueous bedforms. The generality of the equations is appealing. Noting that $\nu\Re$ [$L^2 T^{-1}$] has the dimensions of sediment flux, it can easily be shown that (4) is exactly the well-known Erosion Equation (**Exner, 1925**). The difference here is that the shape of topography is now explicitly accounted for when determining the change in elevation. All terms in the equations can be motivated physically, and the limited number of parameters gives us hope that the coefficients can be related to measurable characteristics of bedform systems. We are just beginning to explore and modify this model to study evolution of subaqueous bedforms, but initial results show that bedform initiation can be treated and that a proper dispersion relationship is recovered.

4. Conclusions

We began by making a case for the importance of transient bedform behavior and recognizing that existing models cannot account for the widely variable kinematics we observe in laboratories and in the field. Bedforms are constantly adjusting, so characterizing the *manner in which they adjust* is at least as important as characterizing asymptotic geometry under relatively steady and uniform conditions. For example, forecasting stage-discharge relationships requires an understanding of rates and pathways of bedform (i.e., roughness) evolution. Accurate descriptions of variability are important if we wish to reconstruct paleohydraulic conditions from stratification produced by bedforms and preserved in sedimentary deposits (e.g., **Paola and Borgman, 1991**). Trains of bedforms are a model example of systems capable of generating their own internal dynamic from steady external forcing. We motivated development of an advection-diffusion approach on phenomenological grounds using the classic bedform migration equation proposed by **Exner (1925)**. Physicists have used Burgers equation, a nonlinear ADE, to describe many interfaces and we attempted to treat bedform evolution using this one-equation model. Several qualitative features of bedforms were reproduced, showing that some bedform behavior is characteristic of generic nonlinear waves. A second equation to explicitly treat sediment flux was sought, which again had already been developed within the physics community in the form of the BCRE equations. Earth scientists and engineers have yet to exploit these equations, which may represent the minimum model necessary to describe bedform initiation and subsequent interactions. The coupled equations (3) and (4) are capable of producing much of the behavior not present with Burgers equation alone. We are optimistic that a BCRE-type model will provide a framework to examine features common to all bedforms – aeolian, fluvial, marine and submarine. Finally, we encourage communication between earth scientists, engineers and physicists, as we are often unaware of theoretical developments in interface science, and physicists are often not aware of the data and insight accumulated from decades of examining this deceptively simple problem.

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References
1. Allen, J.R.L., Phase difference between bed configuration and flow in natural environments, and their geological relevance, *Sedimentology*, *20*, 323-329, 1973.
2. Bouchaud, J.P., M.E. Cates, J. Ravi Prakrash and S.F. Edwards, Hysteresis and metastability in a continuum sandpile model, *Phys. Rev. Lett.*, *74*(11), 1982-1985, 1995.
3. Burgers, J.M., The nonlinear diffusion equation, Reidel, Boston, 1974.
4. Carling, P.A., E. Gölz, H.G. Orr and A. Radecki-Pawlik, The morphodynamics of fluvial sand dunes in the river Rhine, near Mainz, Germany. I. Sedimentology and morphology, *Sedimentology*, *47,* 227-252, 2000.
5. Exner, F.M., Uber die Wechselwirkung zwischen Wasser und Geschiebe in Flüssen, *Sitzungs Bericht Akad. Wiss. Wien*, Mathemathisch-nurtw, Klasse, Abt. Iia, Band 129, 1925 (in German).
6. Fogedby, H.C. Canonical phase-space approach to the noisy Burgers equation, *Phys. Rev. E*, *60*(4), 4950-4953, 1999.
7. Fredsøe, J., Shape and dimensions of stationary dunes in rivers, *Journal of the Hydraulics Division of the American Society of Civil Engineers*, *108*, 932-947, 1982.
8. Medina, E., T. Hwa, M. Kardar and Y. Zhang, Burgers equation with correlated noise : Renormalization-group analysis and applications to directed polymers and interface growth., *Phys. Rev. A*, *39*(6), 3053-3075, 1989.
9. Mohrig, D., Spatial evolution of dunes in a sandy river, Ph. D. dissertation, 119 pp., Univ. of Wash., Seattle, 1994.
10. Mohrig, D and J.D. Smith, Predicting migration rates of subaqueous dunes, *Water Res. Res.*, *32*(10), 3207-3217, 1996.
11. Nagel, K., Particle hopping models and traffic flow theory, *Phys. Rev. E*, *53*(5), 4655-4672, 1996.
12. Paola, C., and Borgman, L., Reconstructing random topography from preserved stratification, *Sedimentology*, *38*, 553-565, 1991.
13. Raudkivi., A.J. and H.H. Witte, Development of bed features, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *116*, 1063-1079, 1990.
14. Terzidis, O., P. Claudin and J.P. Bouchaud, A model for ripple instabilities in granular media., *Euro. Phys. J. B*, *5*, 245-249, 1998.
15. van Rijn, L.C., Sediment transport, Part III: bed forms and alluvial roughness: *Journal of Hydraulic Engineering*, *110*, 1733-1754, 1984.
16. Yalin, M.S., Geometrical properties of sand waves, *J. Hydraul. Div. Am. Soc. Civ. Engrs*, *90*, 105-119, 1964.